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Entanglement in a periodic XX model with long-range interactions

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Abstract

The nearest-neighbor concurrence and block–block entanglement of a period-two XX chain with a transverse field h and a uniform long-range interaction I in the z-direction are studied numerically. It is found that when $I > 0$, the nearest-neighbor concurrence and block–block entanglement are both discontinuous at the critical points, where the system undergoes first-order quantum phase transitions. When the long-range interaction is weak $\left(I < I_{c_1}\right)$, there are two critical points h_{c_1} and h_{c_2}. Between these two points there is a critical region, where the block–block entanglement $S_L \sim \log_2 L$. In one noncritical region $0 \leqslant h \leqslant h_{c_1}$, S_L is equal to a non-zero constant while in the other noncritical region $h \geqslant h_{c_2}$, S_L is always zero. Furthermore, the effect of periodicity on the concurrence and block–block entanglement is discussed.

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1. Introduction

As one interesting simple type of theoretical model, the one-dimensional spin-$\frac{1}{2}$ chains, such as the Heisenberg model, Ising model, XY model and so on, have been studied extensively [1–4]. One of the most frequently studied properties of these systems is the existence of quantum phase transitions (QPTs) at zero temperature [5]. A quantum critical point marks a zero-temperature phase transition between different ground states of a many-body system when we change certain parameters of the system.

Quantum entanglement, as one of the most remarkable traits of quantum systems, has been studied extensively [6]. In the last decade, a great deal of efforts have been focused on the relationship between quantum entanglement and QPTs [7–18]. There are two widely used measures of entanglement for the spin systems. One is concurrence [7] which measures the entanglement between two spins in the spin chain, and the other is block–block entanglement (or von Neumann entropy) [8, 9] which measures the entanglement between a block of L

contiguous spins and the rest of the chain. By using the concurrence, Osterloh *et al* found a universal scaling behavior for the derivative of the entanglement of the uniform anisotropic XY chain in the vicinity of the second-order QPT [10]. Afterward, it is found that the concurrence is discontinuous at the first-order QPT point [11, 12]. By using the von Neumann entropy, Vidal *et al* [9] studied the block entropy S_L of L contiguous spins in the quantum Ising chain in a transverse field. They found that S_L is proportional to $\log_2 L$ at the critical point, while S_L is a constant in the noncritical region.

On the other hand, all of these systems mentioned above are either uniform or disordered with nearest-neighbor or next-nearest-neighbor interactions. It is well known that the properties of periodic spin systems can provide insight into the random spin systems [19–23]. For an anisotropic XY model in a transverse field, it is found that there is only one QPT point if the chain is uniform, but for a periodic or quasiperiodic chain the competition between periodicity and anisotropy gives rise to more QPTs, and the QPTs are all of second order [24–27]. If a uniform long-range interaction among the transverse components of the spins is added to a uniform XX chain, the QPTs become first order [28]. If both periodicity and long-range interaction are taken into account, the model will exhibit complex properties [29].

In this paper, we will study how the competition between the long-range interaction and periodicity affects entanglement. This paper is organized as follows. In section 2, we give the model and the relevant formulae. In section 3, we give the numerical results and their interpretation, and section 4 is a brief conclusion.

2. Model and formulae

We consider a one-dimensional XX model ($S = \frac{1}{2}, N$ sites) with uniform long-range interactions among the z components of the spins in a transverse field. The Hamiltonian is given by [29]

$$H = -\sum_{n=1}^{N} J_n\left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y\right) - h\sum_{n=1}^{N} S_n^z - \frac{I}{N}\sum_{n,m=1}^{N} S_n^z S_m^z, \tag{1}$$

where J_n is the exchange coupling between nearest neighbors, h is a uniform transverse magnetic field, and I is the strength of the long-range interaction. For the uniform system, $J_n = J$ while for the periodic case J_n depends on the site n periodically. In the following, we study a simplest periodic case, i.e. the period-two case, in which, we can take $J_{2n+1} = J$, $J_{2n} = \alpha J$ with $0 \leqslant \alpha < 1$ being the ratio of two nearest-neighbor interactions.

This model can be solved by using the Jordan–Wigner transformation [30] in combination with the Gauss transformation and the steepest descent method [28, 29]. The energy spectrum and the ground-state energy of the system are given by [29]

$$\varepsilon_k^{\pm}(M_z) = (h + 2IM_z) \pm \frac{J}{2}\sqrt{2\alpha\cos(2k) + \alpha^2 + 1} \tag{2}$$

and

$$U(M_z) = -M_z(h + IM_z) - \frac{J}{2\pi}\int_0^{\varphi}\sqrt{1 + \alpha^2 + 2\alpha\cos(2k)}\,\mathrm{d}k, \tag{3}$$

respectively. M_z is the average magnetization and can be solved self-consistently from the equation

$$M_z = \frac{1}{2} - \frac{\varphi}{\pi}, \tag{4}$$

with $\frac{\partial U}{\partial M_z} = 0$ and $\frac{\partial^2 U}{\partial M_z^2} > 0$. Here φ is given by

$$\varphi = \frac{1}{2} \arccos \left[\frac{\left(\frac{2h+4IM_z}{J}\right)^2 - \alpha^2 - 1}{2\alpha} \right], \tag{5}$$

which is derived from $\varepsilon_k^-|_{k=\varphi} = 0$.

In what follows, we use concurrence and block–block entanglement to discuss the entanglement in the ground states of the system.

2.1. Concurrence

Let ρ_{mn} be the reduced density matrix of a pair of spins m and n in the system. The concurrence between spins m and n is defined as

$$C_{mn} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{6}$$

where the quantities $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ are the square roots of the eigenvalues of the operator

$$R_{mn} = \rho_{mn}(\sigma_y \otimes \sigma_y)\rho_{mn}^*(\sigma_y \otimes \sigma_y). \tag{7}$$

The concurrence $C_{mn} = 0$ corresponds to an unentangled state, and $C_{mn} = 1$ corresponds to a completely entangled state. For $[\rho_{mn}, S_z] = 0$ $\left(S_z = \sum_n S_n^z\right)$ the reduced density matrix ρ_{mn} has the form

$$\rho_{mn} = \begin{pmatrix} u^+ & & & \\ & w_1 & z^* & \\ & z & w_2 & \\ & & & u^- \end{pmatrix}. \tag{8}$$

The matrix elements can be written in terms of the correlation functions and the average magnetization M_z as

$$u^\pm = \tfrac{1}{4}\left(1 \pm 4M_z + G_{mn}^{zz}\right), \qquad w_1 = w_2 = \tfrac{1}{4}\left(1 - G_{mn}^{zz}\right), \qquad z = \tfrac{1}{4}\left(G_{mn}^{xx} + G_{mn}^{yy}\right), \tag{9}$$

where $G_{mn}^{zz} = 4\langle S_m^z S_n^z \rangle$, $G_{mn}^{xx} = 4\langle S_m^x S_n^x \rangle$ and $G_{mn}^{yy} = 4\langle S_m^y S_n^y \rangle$. So we can easily find the square roots of the eigenvalues of the operator R_{mn}:

$$\lambda_{1,2} = \pm\sqrt{u^+u^-} \tag{10}$$

and

$$\lambda_{3,4} = z \pm w_1. \tag{11}$$

By taking $g_{mn} = \langle c_m^\dagger c_n \rangle$, the two-point correlations $G_{m,n}^{xx}$, $G_{m,n}^{yy}$ and $G_{m,n}^{zz}$ can be expressed through Wicks theorem by the Jordan–Wigner transformation as

$$u^\pm = \tfrac{1}{4} \pm M_z + M_z^2 - g_{m,n}^2, \tag{12}$$

$$z = g_{m,n} \tag{13}$$

and

$$w_1 = \tfrac{1}{4} - M_z^2 + g_{m,n}^2. \tag{14}$$

For the period-two case,

$$g_{m,m} = M_z + \frac{1}{2}, \tag{15}$$

$$g_{2m-1,2n} = g_{2n,2m-1}^* = \frac{1}{\pi} \int_0^\varphi \frac{\cos(2m-2n)k + \alpha\cos(2m-2n-2)k}{\sqrt{1+\alpha^2+2\alpha\cos 2k}}\,\mathrm{d}k, \tag{16}$$

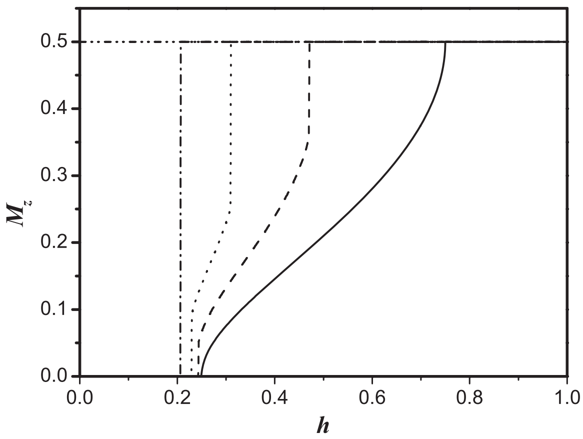

Figure 1. The average magnetization M_z as a function of h. Here $\alpha = 0.5, J = 1$. The solid, dashed, dotted, dashed-dotted and dashed-dotted-dotted curves correspond to $I = 0, 0.3, 0.5, 0.65$ ($\approx I_{c_1}$) and 1.064 ($\approx I_{c_2}$), respectively.

and for $m \neq n$

$$g_{2m-1,2n-1} = g^*_{2n-1,2m-1} = g_{2m,2n} = g^*_{2n,2m} = -\frac{\sin 2(m-n)\varphi}{2(m-n)\pi}. \tag{17}$$

2.2. Block–block entanglement

The entanglement between a block of L contiguous spins and the rest of the chain is defined as

$$S_L \equiv -\mathrm{tr}(\rho_L \log_2 \rho_L). \tag{18}$$

Here ρ_L is the reduced density matrix for L contiguous spins. S_L can be written as [9]

$$S_L = -\sum_{n=1}^{L} [(1-\lambda_n)\log_2(1-\lambda_n) + \lambda_n \log_2 \lambda_n], \tag{19}$$

where λ_n are the eigenvalues of the matrix

$$G_L = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,L} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,L} \\ \cdots & \cdots & \cdots & \cdots \\ g_{L,1} & g_{L,2} & \cdots & g_{L,L} \end{pmatrix}. \tag{20}$$

Here g_{mn} are also defined by equations (15)–(17).

3. Results and discussion

Throughout this section, we take $J = 1$ in calculations without loss of generality.

3.1. Concurrence

The numerical results of average magnetization M_z with different I are given in figure 1 [29]. For the period-two chain, the spins S_{2n} and S_{2n+1} are not equivalent; therefore, the nearest-neighbor concurrence $C_{2n-1,2n}$ and $C_{2n,2n+1}$ are different. But they have similar behavior with

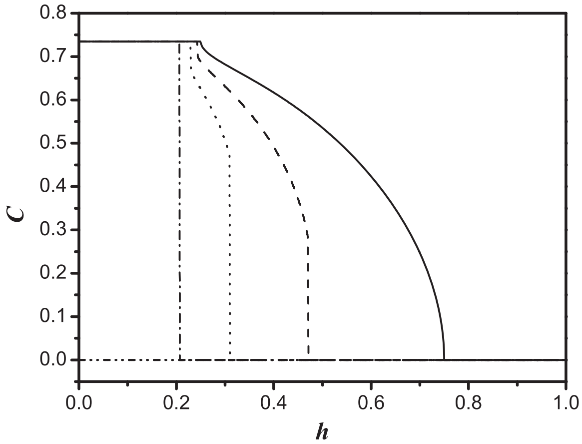

Figure 2. The average nearest-neighbor concurrence C as a function of h. Here $\alpha = 0.5$, $J = 1$. The solid, dashed, dotted, dashed-dotted and dashed-dotted-dotted curves correspond to $I = 0, 0.3, 0.5, 0.65$ ($\approx I_{c_1}$) and 1.064 ($\approx I_{c_2}$), respectively.

respect to the parameters h and I. Therefore, we use the average nearest-neighbor concurrence $C = \frac{1}{2}(C_{2n-1,2n} + C_{2n,2n+1})$ to study the entanglement [26]. Figure 2 shows the average concurrence C as functions of h at different I.

It can easily be found that when the strength of the long-range interaction $I = 0$ (see the solid lines in figures 1 and 2), corresponding to the period-two XX model in a transverse field, there are two critical points at h_{c_1} and h_{c_2} $\left(h_{c_1} < h_{c_2}\right)$. For $h \leqslant h_{c_1}$, $M_z = 0$, the system is in a maximally entangled state corresponding to $C = C_{\max}$. While for $h \geqslant h_{c_2}$, $M_z = \frac{1}{2}$, it is in an untangled state corresponding to $C = C_{\min} = 0$. For $h_{c_1} < h < h_{c_2}$, $0 < M_z < \frac{1}{2}$ and $0 < C < C_{\max}$, h_{c_1} and h_{c_2} mark two second-order phase transitions because on these two critical points the function of M_z is still continuous.

For $I > 0$, the critical behavior of this system is more varied. There are two critical values I_{c_1} and I_{c_2} $\left(I_{c_1} < I_{c_2}\right)$. For $0 < I < I_{c_1}$, there are still two critical points h_{c_1} and h_{c_2} (see the dashed and dotted lines in figures 1 and 2), which both depend on the values of I and α. At these two points, the system undergoes first-order QPTs and M_z is discontinuous. For $h = h_{c_1}$, we have $M_z = M_z^{t_1}$ and 0 with $U(M_z^{t_1}) = U(0)$. And at $h = h_{c_2}$, we have $M_z = M_z^{t_2}$ and $\frac{1}{2}$ with $U\left(M_z^{t_2}\right) = U\left(\frac{1}{2}\right)$. From figure 2, we can see that the concurrence is also discontinuous. At $h = h_{c_1}$, $C = C_{\max}$ and C^{t_1}, whereas $C = C^{t_2}$ and 0 at $h = h_{c_2}$. $M_z^{t_{1,2}}$ and $C^{t_{1,2}}$ are also functions of I and α. As neither M_z nor C is continuous at the critical point h_{c_1} or h_{c_2}, the corresponding QPTs are of first order. As I increases, the difference between h_{c_1} and h_{c_2} decreases. When I goes beyond a critical value I_{c_1} but less than I_{c_2}, (see the dashed-dotted lines in figures 1 and 2) $h_{c_2} = h_{c_1} = h_c$, which means that there is only one critical point. Here $M_z = 0$ and $\frac{1}{2}$ while $C = C_{\max}$ and 0. Similarly, at the critical point h_c, M_z and C are both discontinuous. So the corresponding QPT is also of first order. If $I > I_{c_2}$(see the dashed-dotted-dotted lines in figures 1 and 2) no QPT occurs, and $M_z = \frac{1}{2}$ while $C = 0$ for any h.

Figure 3 gives the phase diagram with the curves of h_c with respect to I [29]. In region 1 $M_z = 0$ while in region 2 $M_z = \frac{1}{2}$. In region 3, which is between the two lines, $0 < M_z < \frac{1}{2}$. This phase diagram can also be used to describe the behavior of C. In region 1, $C = C_{\max}$, which corresponds to a maximally entangled state. In region 2, $C = 0$, which corresponds to an

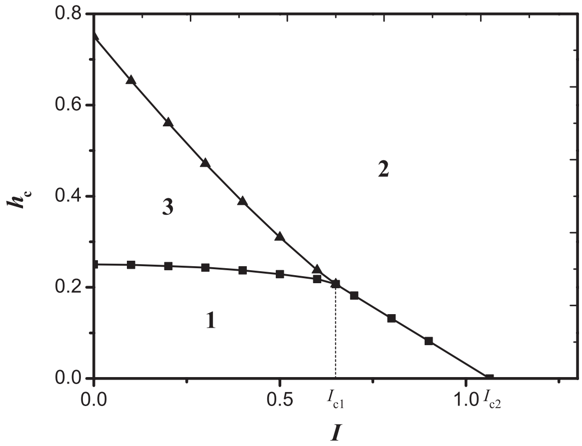

Figure 3. h_c as a function of I. The squares and triangles correspond to h_{c_1} and h_{c_2}, respectively.

unentangled phase. And region 3 corresponds to an entangled state, in which $0 < C < C_{\max}$. Each transition across the solid line is of first order. But the transitions along the h-axis are of second order. It is worth noting that the whole region 3 is critical.

These properties above of QPT are the result of competition between periodicity and long-range interactions (and the transverse field). When $0 < I \ll I_{c_1}$ and the field is weak $\left(0 < h \ll h_{c_1}\right)$, the effect of periodicity is dominant. The average magnetization M_z is always zero. And the ground state is a maximally entangled state corresponding to $C_{\max}$. To see the effect of periodicity, we consider a four-spin system with $\alpha = 0$. It is found that, for $h < \frac{J}{2} - \frac{I}{4}$ and $I < \frac{4J}{3}$, the ground state is

$$\tfrac{1}{2}(|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle).$$

Every two spins connected by J are antiparallel with each other, which corresponds to $M_z = 0$ and $C = \frac{C_{12}+C_{23}}{2} = \frac{1+0.5}{2} = 0.75$. In contrast, both long-range interactions and transverse field always make spins parallel to each other along the direction of the field: $|\uparrow\uparrow \ldots \uparrow\rangle$ and correspond to an unentangled state. The competition between these two opposite effects causes the phase transitions to occur at $h_{c_{1,2}}(I > 0)$, which is less than the case in the absence of long-range interactions $(I = 0)$. And when I is very strong, spins may be outright parallel, which causes the number of critical points to decrease from two to one. And if I is strong enough $\left(I > I_{c_2}\right)$ all the spins are always parallel with each other, and no QPT will occur irrespective of the transverse field h.

We have further calculated M_z (and C) with different α and found their critical behavior to be similar because α $(0 < \alpha < 1)$ only effects on the values of $h_{c_{1,2}}$ and $I_{c_{1,2}}$. So here we just show the results with $\alpha = 0.5$ without loss of generality.

Next we take a look at another effect of periodicity on the concurrence. As we have shown in figure 2, the maximum and minimum of C are both independent of the long-range interaction I. $C_{\min}$ is always equal to zero while

$$C_{\max} = \frac{1}{2} + \left(g_{12}^2 + g_{23}^2\right)\Big|_{\varphi=\frac{\pi}{2}}, \tag{21}$$

which changes with α. For $\alpha = 1$, corresponding to the uniform case, $C_{\max} = \frac{1}{2} + \frac{2}{\pi^2} \approx 0.702$ while for $\alpha = 0$, $C_{\max} = \frac{3}{4}$. We also calculate $C_{\max}$ at different α. The numerical results are shown in figure 4. It is clear that the entanglement between two nearest-neighbor spins in

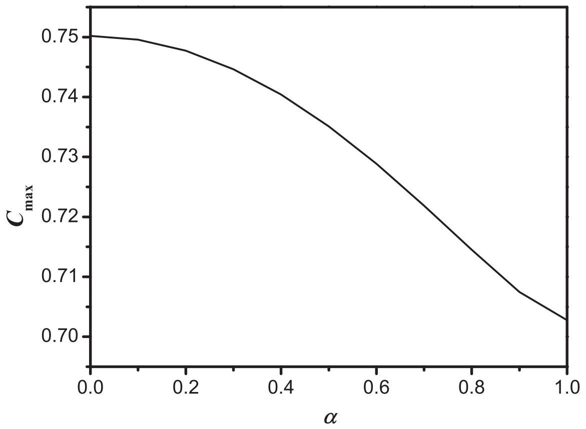

Figure 4. Maximal average concurrence $C_{\max}$ as a function of α with $J = 1$.

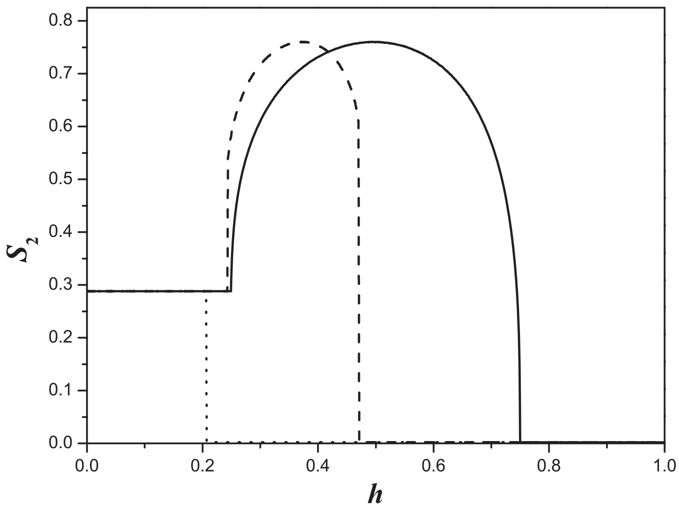

Figure 5. S_2 of the period-two XX model with long-range interactions in a transverse field at zero temperature. $\alpha = 0.5$, $J = 1$. The solid, dashed and dotted curves correspond to $I = 0, 0.3$ and $I_{c_1} \approx 0.65$, respectively.

the periodic system is greater than that in the uniform system when both of them are in the maximally entangled state.

3.2. Block–block entanglement

Concurrence is usually regarded as the entanglement between two spins while block–block entanglement is used to measure the entanglement of a block spin with the rest of the chain.

Figure 5 gives the entanglement of the two spins and the rest part of the system S_2 with $\alpha = 0.5$ at different I. It shows that there are two critical points I_{c_1} and I_{c_2}. When $0 < I < I_{c_1}$, S_2 is discontinuous at two critical points h_{c_1} and h_{c_2}. When $I_{c_1} < I < I_{c_2}$, there is only one critical point with $h_{c_1} = h_{c_2} = h_c$, at which S_2 is also discontinuous. And when

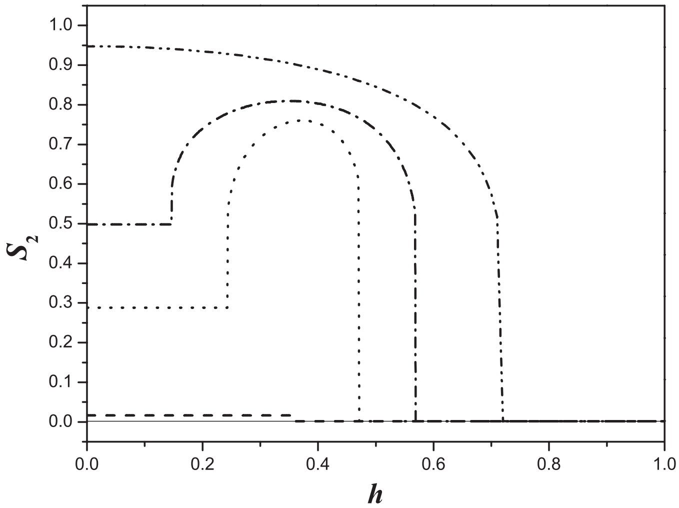

Figure 6. S_2 of the period-two XX model with long-range interactions in a transverse field at zero temperature. $I = 0.3$, $J = 1$. The solid, dashed, dotted, dashed-dotted and dashed-dotted-dotted curves correspond to $\alpha = 0, 0.1, 0.5, 0.7$ and 1.0, respectively.

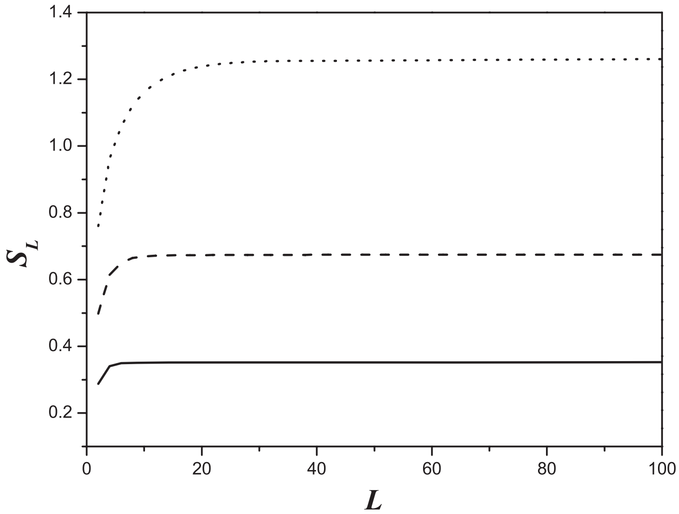

Figure 7. S_L of the period-two XX model with long-range interactions in a transverse field at zero temperature in the noncritical region as a function of L. $I = 0.3$, $h = 0.004$, $J = 1$. The solid, dashed and dotted curves correspond to $\alpha = 0.5, 0.7$ and 0.9, respectively.

$I > I_{c_2}$, S_2 is always zero. We can see that S_2 is a non-zero constant in region 1 of figure 3 and $S_2 = 0$ in region 2 while S_2 varies with h and I in region 3.

Figure 6 gives the curves of S_2 with $I = 0.3$ at different α. It is similar to the nearest-neighbor concurrence; in that S_2 is only determined by α when $h < h_{c_1}$ and $S_2 = 0$ when $h > h_{c_2}$. Here, the value of S_2 in the uniform case is larger than that in the periodic case, which is different from the behavior of nearest-neighbor concurrence.

As is known, the entanglement of a block with a chain S_L in the critical region exhibits a logarithmic divergence for L: $S_L = A \log_2 L + \text{constant}$ with $A = \frac{1}{6}$ for the quantum Ising

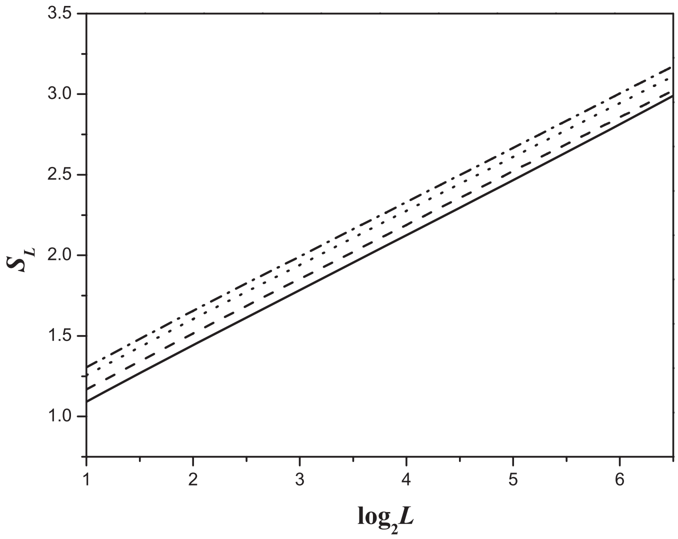

Figure 8. S_L of the period-two XX model with long-range interactions in a transverse field at zero temperature in the critical region as a function of $\log_2 L$. $I = 0.3$, $h = 0.35$ and $J = 1$. The solid, dashed, dotted and dashed-dotted curves correspond to $\alpha = 0.5, 0.7, 0.9$ and 1.0, respectively.

chain and $A = \frac{1}{3}$ for the XX chain with no magnetic field. And S_L is a constant in the noncritical region for the two models [9]. In order to study the scaling behavior, mainly how S_L grows with the block size L in the period-two XX model with long-range interactions in a transverse field, we calculate S_L numerically. Figure 7 shows that in one noncritical region (region 1 in figure 3) S_L is a non-zero constant when the block is large enough. (It can easily be shown that in region 2 of figure 3 S_L is always zero.) And figure 8 shows that in the critical region (region 2 in figure 3) $S_L \propto \log_2 L$ and the slope $A \approx 0.3455 \sim 0.3398$ corresponds to $\alpha = 0.5 \sim 1.0$. This result is in perfect agreement with the earlier result [9].

4. Conclusion

We have studied a period-two XX model with long-range interactions in a transverse field. At zero temperature, the QPTs in this system are more complicated than those in the uniform case. This property is the result of competition between periodicity and long-range interactions. And the behavior of QPTs directly affects the nearest-neighbor concurrence and block–block entanglement of the system. The concurrence takes its extremities not at the critical points $h_{c_{1,2}}$ but in the intervals $0 \leqslant h \leqslant h_{c_1}$ and $h \geqslant h_{c_2}$. The derivative of the entanglement is not existing at first-order QPT points, where the entanglement is discontinuous . The maximum value of concurrence depends on the parameter α. $S_L \sim \log_2 L$ in the critical region and a constant in the noncritical region.

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